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Certain properties of Berry's phases in supersymmetric quantum mechanics: II

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Abstract. In the first paper of our series, we have studied certain crucial mathematical properties of Abelian Berry's phases in two quantum systems with supersymmetrically related Hamiltonians. We follow up such investigation by extending our analysis to non-Abelian Berry's phases in this present study. We present the derivation of an explicit expression for the difference in the relevant non-Abelian Berry's connections. Also we have derived an expression for a connection one form, which contain the non-Abelian Berry's connection, and which is invariant in the two supersymmetrically related quantum systems. To illustrate our findings mentioned above, we take the example of a Hamiltonian expressible in spin quadrupole. In this example, with a particular choice of supersymmetric partner, we show that the two Berry's connection one forms of the two supersymmetrically related systems may be related by gauge transformation.

1. Introduction

The useful concept of geometric phases [1–4] has attracted a lot of attention for the past several years. The natural mathematical explanation for geometric phases can be expressed in terms of the theory of $\mathrm{U}(N)$ fibre bundle [5–7]. Recently we have discussed certain properties of Abelian Berry's phases in supersymmetric quantum mechanics [8]. It is found that the Abelian Berry's phases in two supersymmetrically related quantum systems are not independent but their difference can be derived explicitly. Moreover there is a topological quantity, which contains the Abelian Berry's phase, invariant in the supersymmetrically related quantum systems. Such an invariant topological quantity can be interpreted in the terminology of holonomy and is shown to be corresponding to the essential gauge transformation in a system of spin-$\frac{1}{2}$ particle in a time-varying magnetic field.

We are going to generalize our findings in the Abelian case to non-Abelian case. Instead of a simple phase factor in the Abelian case, an non-Abelian Berry's phase is a $N\times N$ unitary matrix in which N is the dimension of the instantaneous space of degenerate levels. Since Berry's phases result from time evolution of a quantum system, we shall discuss some features of the time-dependent supersymmetric quantum mechanics with degeneracy briefly in section 2. We shall also present the supersymmetric relation between the sets of instantaneous normalized bases for the two supersymmetrically related Hamiltonians, which is essential for our following discussions.

In section 3, we shall first outline the framework of non-Abelian Berry's phase and then derive an explicit expression for the difference in the non-Abelian Berry's connections involving two supersymmetrically related instantaneous space of degenerate levels. Furthermore we shall construct, as we have done in the Abelian case [8], a topological quantity, which behaves as a connection one form, and which is invariant in the two supersymmetrically related quantum systems.

We shall propose an example to illustrate our results obtained in section 4. We adopt a system with spin quadrupole Hamiltonian [9, 10] which possesses a non-Abelian structure with a doubly degenerate sector [10]. Moreover, it is shown that the spin quadrupole Hamiltonian can be factorized in various ways according to supersymmetric quantum mechanics [11-13]. Different ways of factorization yield different pairs of supersymmetric partners. We shall discuss two particular cases among these supersymmetric pairs. Apart from the derivations of the difference in the non-Abelian Berry's connections and the supersymmetric invariant connection, we also find in the first case that the non-Abelian Berry's phase of one member of the supersymmetric partners is trivially equal to the identity operator. However, in the second case, we discuss some gauge properties of the non-Abelian Berry's connections and find that they are related by gauge transformation.

Some relevant discussions and conclusions will be presented in section 5.

2. Time-dependent supersymmetric quantum mechanics with degeneracy

Let us consider a Hamiltonian $\hat{H}_1(\boldsymbol{R})$ which is parametrized by time-dependent parameters $\boldsymbol{R}$ continuously. At any instant, we assume it to have a set of N degenerate levels with positive energy (say $E(\boldsymbol{R})>0$) and the space of these degenerate levels (say $\mathfrak{H}_1(\boldsymbol{R})$) is spanned by a set of instantaneous normalized bases: $\{|\eta_i(\boldsymbol{R})\rangle, i=1,\ldots,N\}$. Furthermore we demand that the Hamiltonian can be factorized as:

$$\hat{H}_1(\boldsymbol{R})=\hat{A}^+(\boldsymbol{R})\hat{A}^-(\boldsymbol{R}) \tag{2.1}$$

where $\hat{A}^-(\boldsymbol{R})$ is some linear operator and $\hat{A}^+(\boldsymbol{R})$ is its adjoint.

The factorization (2.1) allows us to construct another Hamiltonian which is also parametrized by a time-dependent parameter $\boldsymbol{R}$:

$$\hat{H}_2(\boldsymbol{R})=\hat{A}^-(\boldsymbol{R})\hat{A}^+(\boldsymbol{R}). \tag{2.2}$$

The Hamiltonians $\hat{H}_1(\boldsymbol{R})$ and $\hat{H}_2(\boldsymbol{R})$ are said to be supersymmetric partners to each other and should have identical spectra except the zero energy ground state [11-13]. For a set of instantaneous normalized bases $\{|\eta_i(\boldsymbol{R})\rangle, i=1,\ldots,N\}$, we can construct a set of instantaneous normalized kets $\{|\zeta_i(\boldsymbol{R})\rangle, i=1,\ldots,N\}$ which are defined by:

$$\hat{A}^-(\boldsymbol{R})|\eta_i(\boldsymbol{R})\rangle=\sqrt{E(\boldsymbol{R})}\,|\zeta_i(\boldsymbol{R})\rangle \tag{2.3a}$$

or conversely,

$$\hat{A}^+(\boldsymbol{R})|\zeta_i(\boldsymbol{R})\rangle=\sqrt{E(\boldsymbol{R})}\,|\eta_i(\boldsymbol{R})\rangle. \tag{2.3b}$$

Obviously, $|\zeta_i(\boldsymbol{R})\rangle$ are orthogonal to each other because of the orthogonality of $|\eta_i(\boldsymbol{R})\rangle$ and can be shown to be eigenkets of $\hat{H}_2(\boldsymbol{R})$ with energy eigenvalue $E(\boldsymbol{R})$. Therefore $\{|\zeta_i(\boldsymbol{R})\rangle, i=1,\ldots,N\}$ span a N dimensional space of degenerate levels (say $\mathfrak{H}_2(\boldsymbol{R})$) of $\hat{H}_2(\boldsymbol{R})$. Moreover, any instantaneous eigenket of $\hat{H}_2(\boldsymbol{R})$ with energy eigenvalue $E(\boldsymbol{R})$ must be a linear combination of $|\zeta_i(\boldsymbol{R})\rangle$ and hence is included in $\mathfrak{H}_2(\boldsymbol{R})$ for the following argument. Assume that $|\phi_2(\boldsymbol{R})\rangle$ to be an instantaneous eigenket of $\hat{H}_2(\boldsymbol{R})$ with energy eigenvalue $E(\boldsymbol{R})$ but is not included in $\mathfrak{H}_2(\boldsymbol{R})$. Analogous to (2.3$b$) we can construct an instantaneous eigenket, say $|\phi_1(\boldsymbol{R})\rangle$, of $\hat{H}_1(\boldsymbol{R})$ with energy eigenvalue $E(\boldsymbol{R})$. However, it is easily shown that $|\phi_1(\boldsymbol{R})\rangle$ does not belong to $\mathfrak{H}_1(\boldsymbol{R})$ which contradicts the fact that the space of degenerate levels of $\hat{H}_1(\boldsymbol{R})$ with energy $E(\boldsymbol{R})$ at any instant is a N dimensional space spanned by $\{|\eta_i(\boldsymbol{R})\rangle, i=1, \ldots, N\}$.

Consequently, because of the presence of supersymmetry between $\hat{H}_1(\boldsymbol{R})$ and $\hat{H}_2(\boldsymbol{R})$, the existence of N dimensional spaces of degenerate levels $(\mathfrak{H}_1(\boldsymbol{R}))$ of $\hat{H}_1(\boldsymbol{R})$ implies the existence of N dimensional spaces of degenerate levels $(\mathfrak{H}_2(\boldsymbol{R}))$ of $\hat{H}_2(\boldsymbol{R})$ and their corresponding sets of instantaneous normalized bases, $\{|\eta_i(\boldsymbol{R})\rangle, i=1, \ldots, N\}$ and $\{|\zeta_i(\boldsymbol{R})\rangle, i=1, \ldots, N\}$, are transformed into each other as stated in (2.3*a*, *b*).

We would like to remark that we have described time-dependent supersymmetric quantum mechanics in both non-degenerate [8] and degenerate cases. In both cases, we have assumed that the instantaneous energy eigenvalue $E(\boldsymbol{R})$ is always positive and the Hamiltonian $\hat{H}_1(\boldsymbol{R})$ can be factorized as in (2.1) throughout the evolution. As long as these two assumptions hold, the construction of $\hat{H}_2(\boldsymbol{R})$ (2.2) becomes possible and the corresponding energy eigenkets are related by (2.3*a*, *b*) which make our discussions on the properties of Berry's phases in the following sections feasible.

3. Non-Abelian Berry's phases in supersymmetric quantum mechanics

We consider a quantum system governed by $\hat{H}_1(\boldsymbol{R})$ which evolves adiabatically. We denote the wavefunction of the system by $|\psi_j(\boldsymbol{R})\rangle$ at any instant and assume the initial condition:

$$|\psi_j(\boldsymbol{R}_0)\rangle = |\eta_j(\boldsymbol{R}_0)\rangle \tag{3.1}$$

where $\boldsymbol{R}_0$ is the value of the parameter at time $t=0$. Under the adiabatic assumption, $|\psi_j(\boldsymbol{R})\rangle$ may be expanded in terms of $|\eta_i(\boldsymbol{R})\rangle$:

$$|\psi_j(\boldsymbol{R})\rangle = \sum_i |\eta_i(\boldsymbol{R})\rangle \mathcal{U}_{ij}(t) \tag{3.2}$$

where $\boldsymbol{R}$ is the value of the parameter at time t. Without loss of generality, we renormalize the instantaneous energy such that $E(\boldsymbol{R})=0$. Substituting (3.2) into the Schrödinger equation: $\mathrm{i}(\partial/\partial t)|\psi_j(\boldsymbol{R})\rangle = \hat{H}_1(\boldsymbol{R})|\psi_j(\boldsymbol{R})\rangle$, we find:

$$\dot{\mathcal{U}}_{ij}(t) = -\sum_k \langle \eta_i(\boldsymbol{R})|\dot{\eta}_k(\boldsymbol{R})\rangle \mathcal{U}_{kj}(t) \tag{3.3}$$

where the derivative is with respect to time t. By introduction of the gauge potential $\langle \eta_i(\boldsymbol{R})|\nabla_R \eta_j(\boldsymbol{R})\rangle$ and the corresponding connection one form $\mathcal{A}(\boldsymbol{R})$ defined as:

$$\mathcal{A}_{ij}(\boldsymbol{R}) = \langle \eta_i(\boldsymbol{R})|\nabla_R \eta_j(\boldsymbol{R})\rangle \cdot \mathrm{d}\boldsymbol{R} \tag{3.4}$$

with initial condition induced by (3.1) i.e.

$$\mathcal{U}(0) = I \tag{3.5}$$

the solution of (3.3) is given by:

$$\mathcal{U}(t) = \mathcal{P} \exp\left[-\int \mathcal{A}(\boldsymbol{R})\right] \tag{3.6}$$

in which $\mathcal{P}$ stands for path-ordered product and hence $\mathcal{U}(t)$ is path dependent and is not a well-defined function of parameter $\boldsymbol{R}$.

Similar arguments can be applied to the quantum system governed by $\hat{H}_2(\boldsymbol{R})$. Under the adiabatic approximation and the initial condition:

$$|\phi_j(\boldsymbol{R}_0)\rangle = |\zeta_j(\boldsymbol{R}_0)\rangle \tag{3.7}$$

in which $|\zeta_j(\boldsymbol{R}_0)\rangle$ is supersymmetrically related to $|\eta_j(\boldsymbol{R}_0)\rangle$ in view of (2.3), the wavefunction of the system at any instant can be expressed as:

$$|\phi_j(\boldsymbol{R})\rangle = \sum_i |\zeta_i(\boldsymbol{R})\rangle \mathbb{U}_{ij}(t) \tag{3.8}$$

where we have chosen the states $|\zeta_i(\boldsymbol{R})\rangle$ which are defined in (2.3) as the instantaneous bases for the space of degenerate levels.

The unitary matrix $\mathbb{U}$ should satisfy the differential equation analogous to (3.3):

$$\dot{\mathbb{U}}_{ij}(t) = -\sum_k \langle \zeta_i(\boldsymbol{R})|\dot{\zeta}_k(\boldsymbol{R})\rangle \mathbb{U}_{kj}(t) \tag{3.9}$$

and its solution is then:

$$\mathbb{U}(t) = \mathscr{P} \exp\left[-\int \mathbb{A}(\boldsymbol{R})\right] \tag{3.10}$$

where $\mathbb{A}$ is the connection one form analogous to $\mathscr{A}(\boldsymbol{R})$ defined as:

$$\mathbb{A}_{ij}(\boldsymbol{R}) = \langle \zeta_i(\boldsymbol{R})|\nabla_R \zeta_j(\boldsymbol{R})\rangle \cdot \mathrm{d}\boldsymbol{R}. \tag{3.11}$$

The expressions (3.6) and (3.10) of $\mathscr{U}(t)$ and $\mathbb{U}(t)$ are known as non-Abelian Berry's phases [2] and they are indeed not independent because of the existence of supersymmetry between the systems. To see this point, let us first consider the operation of ∇_R on both sides of (2.3), we arrive at:

$$\begin{aligned}(\nabla_R \hat{A}^-(\boldsymbol{R}))|\eta_j(\boldsymbol{R})\rangle &+ \hat{A}^-(\boldsymbol{R})|\nabla_R \eta_j(\boldsymbol{R})\rangle \\ &= (\nabla_R \sqrt{E(\boldsymbol{R})})|\zeta_j(\boldsymbol{R})\rangle + \sqrt{E(\boldsymbol{R})}\,|\nabla_R \zeta_j(\boldsymbol{R})\rangle \end{aligned} \tag{3.12a}$$

$$\begin{aligned}(\nabla_R \hat{A}^+(\boldsymbol{R}))|\zeta_j(\boldsymbol{R})\rangle &+ \hat{A}^+(\boldsymbol{R})|\nabla_R \zeta_j(\boldsymbol{R})\rangle \\ &= (\nabla_R \sqrt{E(\boldsymbol{R})})|\eta_j(\boldsymbol{R})\rangle + \sqrt{E(\boldsymbol{R})}\,|\nabla_R \eta_j(\boldsymbol{R})\rangle. \end{aligned} \tag{3.12b}$$

We then operate both equations by $\langle \zeta_i(\boldsymbol{R})|$ and $\langle \eta_i(\boldsymbol{R})|$ respectively:

$$\begin{aligned}\langle \zeta_i(\boldsymbol{R})|(\nabla_R \hat{A}^-(\boldsymbol{R}))|\eta_j(\boldsymbol{R})\rangle &+ \sqrt{E(\boldsymbol{R})}\,\langle \eta_i(\boldsymbol{R})|\nabla_R \eta_j(\boldsymbol{R})\rangle \\ &= \nabla_R \sqrt{E(\boldsymbol{R})}\,\delta_{ij} + \sqrt{E(\boldsymbol{R})}\,\langle \zeta_i(\boldsymbol{R})|\nabla_R \zeta_j(\boldsymbol{R})\rangle \end{aligned} \tag{3.13a}$$

$$\begin{aligned}\langle \eta_i(\boldsymbol{R})|(\nabla_R \hat{A}^+(\boldsymbol{R}))|\zeta_j(\boldsymbol{R})\rangle &+ \sqrt{E(\boldsymbol{R})}\,\langle \zeta_i(\boldsymbol{R})|\nabla_R \zeta_j(\boldsymbol{R})\rangle \\ &= \nabla_R \sqrt{E(\boldsymbol{R})}\,\delta_{ij} + \sqrt{E(\boldsymbol{R})}\,\langle \eta_i(\boldsymbol{R})|\nabla_R \eta_j(\boldsymbol{R})\rangle. \end{aligned} \tag{3.13b}$$

In (3.12) and (3.13), the instantaneous energy $E(\boldsymbol{R})$ is not zero although we have renormalized it to be zero to obtain (3.6) and (3.10). In fact, the renormalization is used to eliminate the dynamical 'phase' and simplify the calculations. However we must emphasize that the topological 'phase' is given in the form of (3.6) and (3.10) no matter whether $E(\boldsymbol{R}) = 0$ is valid.

We define the difference of the gauge potentials $\Delta_{ij}(\boldsymbol{R}) \equiv \langle \eta_i(\boldsymbol{R})|\nabla_R \eta_j(\boldsymbol{R})\rangle - \langle \zeta_i(\boldsymbol{R})|\nabla_R \zeta_j(\boldsymbol{R})\rangle$ and obtain its expression using (3.13):

$$\Delta_{ij}(\boldsymbol{R}) = \frac{\langle \eta_i(\boldsymbol{R})|(\nabla_R \hat{A}^+(\boldsymbol{R}))|\zeta_j(\boldsymbol{R})\rangle - \langle \zeta_i(\boldsymbol{R})|(\nabla_R \hat{A}^-(\boldsymbol{R}))|\eta_j(\boldsymbol{R})\rangle}{2\sqrt{E(\boldsymbol{R})}}. \tag{3.14}$$

Moreover, with the help of (3.13), we also find that there is an equality of topological quantities including the gauge potentials:

$$\langle\eta_i(\boldsymbol{R})|\nabla_R\eta_j(\boldsymbol{R})\rangle+\langle\eta_i(\boldsymbol{R})|\hat{A}^+(\boldsymbol{R})(\nabla_R\hat{A}^-(\boldsymbol{R}))|\eta_j(\boldsymbol{R})\rangle/(2E(\boldsymbol{R}))$$
$$=\langle\zeta_i(\boldsymbol{R})|\nabla_R\zeta_j(\boldsymbol{R})\rangle+\langle\zeta_i(\boldsymbol{R})|\hat{A}^-(\boldsymbol{R})(\nabla_R\hat{A}^+(\boldsymbol{R}))|\zeta_j(\boldsymbol{R})\rangle/(2E(\boldsymbol{R})). \tag{3.15}$$

The equality (3.15) shows us that the topological quantities on both sides do not share the same form. However, because the role of the Hamiltonians can be exchanged, the Hamiltonians may be factorized in another way:

$$\hat{H}_2(\boldsymbol{R})=\hat{B}^+(\boldsymbol{R})\hat{B}^-(\boldsymbol{R}) \tag{3.16a}$$
$$\hat{H}_1(\boldsymbol{R})=\hat{B}^-(\boldsymbol{R})\hat{B}^+(\boldsymbol{R}) \tag{3.16b}$$

where $\hat{B}^{\pm}(\boldsymbol{R})$ are some linear operators different from $\hat{A}^{\pm}(\boldsymbol{R})$ in general. Moreover operators $\hat{B}^{\pm}(\boldsymbol{R})$ induce transformations analogous to (2.3*a, b*):

$$\hat{B}^+(\boldsymbol{R})|\eta_i(\boldsymbol{R})\rangle=\sqrt{E(\boldsymbol{R})}\,|\xi_i(\boldsymbol{R})\rangle \tag{3.17a}$$
$$\hat{B}^-(\boldsymbol{R})|\xi_i(\boldsymbol{R})\rangle=\sqrt{E(\boldsymbol{R})}\,|\eta_i(\boldsymbol{R})\rangle. \tag{3.17b}$$

$|\xi_i(\boldsymbol{R})\rangle$ are eigenkets of $\hat{H}_2(\boldsymbol{R})$ with energy $E(\boldsymbol{R})$ and hence must be related with $|\zeta_i(\boldsymbol{R})\rangle$ by an unitary transformation:

$$|\xi_i(\boldsymbol{R})\rangle=\hat{\Gamma}|\zeta_i(\boldsymbol{R})\rangle \tag{3.18}$$

where $\hat{\Gamma}$ is an unitary operator and must commute with $\hat{H}_2(\boldsymbol{R})$.

In fact (3.18) is true for any energy eigenket of $\hat{H}_2(\boldsymbol{R})$, therefore the most general relations between $\hat{B}^{\mp}(\boldsymbol{R})$ and $\hat{A}^{\pm}(\boldsymbol{R})$ are then:

$$\hat{B}^-(\boldsymbol{R})=\hat{A}^+(\boldsymbol{R})\hat{\Gamma}^+ \tag{3.19a}$$
$$\hat{B}^+(\boldsymbol{R})=\hat{\Gamma}\hat{A}^-(\boldsymbol{R}). \tag{3.19b}$$

Because of (3.16*a, b*), we can also obtain an equality analogous to (3.15) but in terms of $\hat{B}^{\pm}(\boldsymbol{R})$:

$$\langle\eta_i(\boldsymbol{R})|\nabla_R\eta_j(\boldsymbol{R})\rangle+\langle\eta_i(\boldsymbol{R})|\hat{B}^-(\boldsymbol{R})(\nabla_R\hat{B}^+(\boldsymbol{R}))|\eta_j(\boldsymbol{R})\rangle/(2E(\boldsymbol{R}))$$
$$=\langle\xi_i(\boldsymbol{R})|\nabla_R\xi_j(\boldsymbol{R})\rangle+\langle\xi_i(\boldsymbol{R})|\hat{B}^+(\boldsymbol{R})(\nabla_R\hat{B}^-(\boldsymbol{R}))|\xi_j(\boldsymbol{R})\rangle/(2E(\boldsymbol{R})). \tag{3.20}$$

Moreover by using (3.19*a, b*), we get:

$$\langle\eta_i(\boldsymbol{R})|\hat{B}^-(\boldsymbol{R})(\nabla_R\hat{B}^+(\boldsymbol{R}))|\eta_j(\boldsymbol{R})\rangle/(2E(\boldsymbol{R}))$$
$$=\langle\eta_i(\boldsymbol{R})|\hat{A}^+(\boldsymbol{R})(\nabla_R\hat{A}^-(\boldsymbol{R}))|\eta_j(\boldsymbol{R})\rangle/(2E(\boldsymbol{R}))$$
$$+\langle\zeta_i(\boldsymbol{R})|\hat{\Gamma}^+(\nabla_R\hat{\Gamma})|\zeta_j(\boldsymbol{R})\rangle/2 \tag{3.21a}$$

and

$$\langle\xi_i(\boldsymbol{R})|\hat{B}^+(\boldsymbol{R})(\nabla_R\hat{B}^-(\boldsymbol{R}))|\xi_j(\boldsymbol{R})\rangle/(2E(\boldsymbol{R}))$$
$$=\langle\zeta_i(\boldsymbol{R})|\hat{A}^-(\boldsymbol{R})(\nabla_R\hat{A}^+(\boldsymbol{R}))|\zeta_j(\boldsymbol{R})\rangle/(2E(\boldsymbol{R}))$$
$$+\langle\zeta_i(\boldsymbol{R})|\hat{\Gamma}^+(\nabla_R\hat{\Gamma})|\zeta_j(\boldsymbol{R})\rangle/2. \tag{3.21b}$$

In particular, we consider an unitary operator $\hat{\Gamma}$ which vanishes under the operation of ∇_R such that $\nabla_R\hat{\Gamma}\equiv 0$ and hence the last term in both (3.21*a, b*) is identical zero. By combining (3.15) and (3.20), the equality becomes:

$$\langle\eta_i(\boldsymbol{R})|\nabla_R\eta_j(\boldsymbol{R})\rangle+\langle\eta_i(\boldsymbol{R})|\hat{A}^+(\boldsymbol{R})(\nabla_R\hat{A}^-(\boldsymbol{R}))|\eta_j(\boldsymbol{R})\rangle/(2E(\boldsymbol{R}))$$
$$=\langle\xi_i(\boldsymbol{R})|\nabla_R\xi_j(\boldsymbol{R})\rangle+\langle\xi_i(\boldsymbol{R})|\hat{B}^+(\boldsymbol{R})(\nabla_R\hat{B}^-(\boldsymbol{R}))|\xi_j(\boldsymbol{R})\rangle/(2E(\boldsymbol{R})). \tag{3.22}$$

Clearly, both sides in (3.22) share the same form and we conclude that the toplogical quantity, $\mu_1(\boldsymbol{R})$:

$$\mu_{lij}(\boldsymbol{R}) = \langle \eta_i(\boldsymbol{R})|\nabla_R \eta_j(\boldsymbol{R})\rangle + \langle \eta_i(\boldsymbol{R})|\hat{A}_l^+(\boldsymbol{R})(\nabla_R \hat{A}_l^-(\boldsymbol{R}))|\eta_j(\boldsymbol{R})\rangle/(2E(\boldsymbol{R})) \tag{3.23}$$

is invariant in the two supersymmetrically related systems whose Hamiltonians are written in the forms: $\hat{H}_l(\boldsymbol{R}) = \hat{A}_l^+(\boldsymbol{R})\hat{A}_l^-(\boldsymbol{R})$, where $l = 1$ or 2.

As done in [8], we would like to construct an expression of connection one form which is invariant in the two supersymmetrically related systems. It is well known that a connection one form is required to have values in the Lie algebra of the structure group of the principal fibre bundle. In our discussions, the structure group is U(N) and the invariant quantity we obtain in (3.22) may not satisfy the requirement. Fortunately, it can easily be shown that an invariant quantity that satisfy the requirement is simply, $\lambda_l(\boldsymbol{R})$:

$$\lambda_{lij}(\boldsymbol{R}) = \mu_{lij}(\boldsymbol{R}) - (\nabla_R \sqrt{E(\boldsymbol{R})}/2\sqrt{E(\boldsymbol{R})})\delta_{ij}. \tag{3.24}$$

Moreover, with a different choice of instantaneous normalized bases

$$|\eta_i'(\boldsymbol{R})\rangle = \sum_j |\eta_j(\boldsymbol{R})\rangle \omega_{ji}(\boldsymbol{R}) \tag{3.25a}$$

or

$$|\xi_i'(\boldsymbol{R})\rangle = \sum_j |\xi_j(\boldsymbol{R})\rangle \omega_{ji}(\boldsymbol{R}) \tag{3.25b}$$

where $\omega(\boldsymbol{R})$ is an $N \times N$ unitary matrix, we can show that the invariant quantity: $\lambda_l(\boldsymbol{R}) \cdot \mathrm{d}\boldsymbol{R}$ transforms as a usual gauge potential:

$$\lambda_l(\boldsymbol{R}) \cdot \mathrm{d}\boldsymbol{R} \to \omega^+(\boldsymbol{R})[\lambda_l(\boldsymbol{R}) \cdot \mathrm{d}\boldsymbol{R}]\omega(\boldsymbol{R}) + \omega^+(\boldsymbol{R})\,\mathrm{d}\omega(\boldsymbol{R}) \tag{3.26}$$

and the difference $\Delta(\boldsymbol{R}) \cdot \mathrm{d}\boldsymbol{R}$ obtained in (3.14) transforms with a covariant manner with (3.26) but without the inhomogeneous term, i.e.

$$\Delta(\boldsymbol{R}) \cdot \mathrm{d}\boldsymbol{R} \to \omega^+(\boldsymbol{R})[\Delta(\boldsymbol{R}) \cdot \mathrm{d}\boldsymbol{R}]\omega(\boldsymbol{R}). \tag{3.27}$$

Finally we conclude, in accordance with (3.22) and (3.27), that there is a connection one form $\lambda_l(\boldsymbol{R}) \cdot \mathrm{d}\boldsymbol{R}$ which is invariant in the two supersymmetrically related systems analogous to that obtained in the Abelian case [8].

4. Example

In this section, we discuss a quantum system described by the Hamiltonian in the form of spin quadrupole:

$$\hat{H} = (\hat{S} \cdot \boldsymbol{B})^2 \tag{4.1}$$

in which the magnetic field varies adiabatically and is parametrized by polar angles (ϑ, φ):

$$\boldsymbol{B} = |\boldsymbol{B}|(\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta). \tag{4.2}$$

The effect of Berry's phase in such system was investigated experimentally [9] and the features of the non-Abelian Berry's phase in this system have been discussed theoretically in details recently [10]. Let us start our discussion by expressing it in terms of polar angles ϑ and φ:

$$\begin{aligned}\hat{H}(\vartheta, \varphi) &= (S \cdot \boldsymbol{B})^2 \\ &= (\hat{S}_x \sin\vartheta\cos\varphi + \hat{S}_y \sin\vartheta\sin\varphi + \hat{S}_z\cos\vartheta)^2|\boldsymbol{B}|^2.\end{aligned} \tag{4.3}$$

The parameter space in this example is simply 2-sphere and it should be noted that the polar angles (ϑ, φ) is not everywhere defined on 2-sphere.

Alternately (4.2) may be rewritten to be:

$$\hat{H}(\vartheta, \varphi) = \exp(-\mathrm{i}\varphi\hat{S}_z)\exp(-\mathrm{i}\vartheta\hat{S}_y)\hat{S}_z^2\exp(\mathrm{i}\vartheta\hat{S}_y)\exp(\mathrm{i}\varphi\hat{S}_z) \tag{4.4}$$

in which we set $|\boldsymbol{B}| = 1$ for simplicity. The instantaneous eigenstate of $\hat{H}(\vartheta, \varphi)$ is then given by [2, 10]:

$$|\eta_i(\vartheta, \varphi)\rangle = \exp(-\mathrm{i}\varphi\hat{S}_z)\exp(-\mathrm{i}\vartheta\hat{S}_y)|\mathrm{i}\rangle \tag{4.5}$$

where $|\mathrm{i}\rangle$ can be taken to be the eigenstates of $\hat{S}_z$. Because the eigenstates with eigenvalues $\pm m$ form a doubly degenerate sector, the dimension of the degenerate levels is 2. Moreover it is shown that non-Abelian structure arises only for $|m| = \frac{1}{2}$ [10] and we will restrict our following discussions for this case. Using (4.5) we obtain the non-Abelian connection one form defined in (3.4):

$$\mathscr{A}(\vartheta, \varphi) = (-\mathrm{i})\left[\left(\cos\vartheta\frac{\sigma_3}{2} - \alpha\sin\vartheta\frac{\sigma_1}{2}\right)\mathrm{d}\varphi + \alpha\frac{\sigma_2}{2}\mathrm{d}\vartheta\right] \tag{4.6}$$

where $\alpha \equiv S + \frac{1}{2}$ and σ_k are the standard Pauli matrices. Moreover the corresponding gauge field $\mathscr{F} = \mathrm{d}\mathscr{A} + \mathscr{A} \wedge \mathscr{A}$ is given by:

$$\mathscr{F}(\vartheta, \varphi) = (-\mathrm{i})(\alpha^2 - 1)\frac{\sigma_3}{2}\mathrm{d}\Omega \tag{4.7}$$

with $\mathrm{d}\Omega = \sin\vartheta\,\mathrm{d}\vartheta \wedge \mathrm{d}\varphi$.

Identifying $\hat{H}(\vartheta, \varphi)$ with $\hat{H}_1$ mentioned in the previous section, we can factorize the Hamiltonian:

$$\hat{H}_1(\vartheta, \varphi) = \hat{A}^+(\vartheta, \varphi)\hat{A}^-(\vartheta, \varphi) \tag{4.8}$$

with

$$\hat{A}^+(\vartheta, \varphi) = \exp(-\mathrm{i}\varphi\hat{S}_z)\exp(-\mathrm{i}\vartheta\hat{S}_y)\hat{S}_z\exp(\mathrm{i}\vartheta'\hat{S}_y)\exp(\mathrm{i}\varphi'\hat{S}_z) \tag{4.9a}$$

$$\hat{A}^-(\vartheta, \varphi) = \exp(-\mathrm{i}\varphi'\hat{S}_z)\exp(-\mathrm{i}\vartheta'\hat{S}_y)\hat{S}_z\exp(\mathrm{i}\vartheta\hat{S}_y)\exp(\mathrm{i}\varphi\hat{S}_z) \tag{4.9b}$$

where (ϑ', φ') is another pair of polar angles and the supersymmetric partner $\hat{H}_2(\vartheta, \varphi)$ is given by:

$$\begin{aligned}\hat{H}_2(\vartheta, \varphi) &= \hat{A}^-(\vartheta, \varphi)\hat{A}^+(\vartheta, \varphi)\\ &= \exp(-\mathrm{i}\varphi'\hat{S}_z)\exp(-\mathrm{i}\vartheta'\hat{S}_y)\hat{S}_z^2\exp(\mathrm{i}\vartheta'\hat{S}_y)\exp(\mathrm{i}\varphi'\hat{S}_z).\end{aligned} \tag{4.10}$$

Note that (4.10) is the spin quadrupole Hamiltonian with magnetic field which orientates along the direction $(\sin\vartheta'\cos\varphi', \sin\vartheta'\sin\varphi', \cos\vartheta')$. Therefore any pair of Hamiltonians having the form (4.1) are supersymmetrically related no matter what orientations of the magnetic fields are. However, in the following discussions, we will consider two particular cases in which (ϑ', φ') are functions of (ϑ, φ) and thus our arguments in the previous section become feasible.

4.1. $(\vartheta', \varphi') = (0, 0)$

Firstly, we consider the simplest case in which $\vartheta' = \varphi' = 0$ such that $\hat{H}_2(\vartheta, \varphi)$ in (4.10) is then:

$$\hat{H}_2(\vartheta, \varphi) = \hat{S}_z^2. \tag{4.1.1}$$

It is noticed that $\hat{H}_2$ is actually independent of the polar angles (ϑ, φ) and the magnetic field is pointing towards the z direction forever. The instantaneous eigenstates of $\hat{H}_2$ are also eigenstates of $\hat{S}_z$ with eigenvalues $\pm\frac{1}{2}$ and denoted by $|\pm\frac{1}{2}\rangle$. The connection one form defined in (3.11) and its corresponding gauge field $\mathbb{F} = \mathrm{d}\mathbb{A} + \mathbb{A} \wedge \mathbb{A}$ are given by:

$$\mathbb{A} = \mathbb{F} = 0. \tag{4.1.2}$$

Because $\mathbb{A}(\vartheta, \varphi)$ is identical to zero in this example, the Berry's phase given by (3.10) is no longer non-Abelian but is equal to the identity operator I.

We can also evaluate the quantities mentioned in (3.14), (3.23) and (3.24):

$$\Delta(\vartheta, \varphi)\cdot \mathrm{d}(\vartheta, \varphi) = \mathscr{A}(\vartheta, \varphi) \tag{4.1.3}$$

$$\mu_l(\vartheta, \varphi)\cdot \mathrm{d}(\vartheta, \varphi) = \lambda_l(\vartheta, \varphi)\cdot \mathrm{d}(\vartheta, \varphi) = \mathscr{A}(\vartheta, \varphi)/2 \tag{4.1.4}$$

with $l = 1$ or 2. The equality between $\mu_l(\vartheta, \varphi)$ and $\lambda_l(\vartheta, \varphi)$ in (4.1.4) arises due to the fact that the instantaneous energy $E(\vartheta, \varphi)$ is independent on the polar angles (ϑ, φ).

It is worth mentioning that the connection one forms $\mathscr{A}$ and $\mathbb{A}$ cannot be related by gauge transformation in this particular case. To see this point, we consider the gauge fields $\mathscr{F}$ and $\mathbb{F}$ which are gauge covariant. However, $\mathscr{F}$ is not zero while $\mathbb{F}$ is zero as stated in (4.1.2).

4.2. $(\vartheta, \varphi) = (-\vartheta, -\varphi)$

Secondly, we consider the case in which $(\vartheta, \varphi) = (-\vartheta, -\varphi)$ such that $\hat{H}_2(\vartheta, \varphi)$ in (4.10) is then:

$$\hat{H}_2(\vartheta, \varphi) = \exp(\mathrm{i}\varphi\hat{S}_z)\exp(\mathrm{i}\vartheta\hat{S}_y)\hat{S}_z^2\exp(-\mathrm{i}\vartheta\hat{S}_y)\exp(-\mathrm{i}\varphi\hat{S}_z). \tag{4.2.1}$$

The magnetic field involved in the system described by $\hat{H}_2(\vartheta, \varphi)$ orientates in a direction opposite to that of $\hat{H}_1(\vartheta, \varphi)$ and its instantaneous eigenstate can be obtained by using (2.3):

$$\begin{aligned}|\zeta_i(\vartheta, \varphi)\rangle &= \hat{A}^-(\vartheta, \varphi)|\eta_i(\vartheta, \varphi)\rangle/\sqrt{E(\boldsymbol{R})} \\ &= \exp(\mathrm{i}\varphi\hat{S}_z)\exp(\mathrm{i}\vartheta\hat{S}_y)|\mathrm{i}\rangle\end{aligned} \tag{4.2.2}$$

where we have used (4.5) and (4.9b). It should be recalled that we restrict our discussion for $|m| = \frac{1}{2}$ and therefore $|\mathrm{i}\rangle$ represent the eigenstates of $\hat{S}_z$: $|\pm\frac{1}{2}\rangle$.

The difference of the connection one forms can be calculated according to (3.14):

$$\Delta(\vartheta, \varphi)\cdot \mathrm{d}(\vartheta, \varphi) = (-\mathrm{i})(\cos\vartheta\sigma_3\,\mathrm{d}\varphi + \alpha\sigma_2\,\mathrm{d}\vartheta). \tag{4.2.3}$$

Note that we have already obtained the states $|\zeta_i(\vartheta, \varphi)\rangle$ in (4.2.2). By direct substitution into (3.11) we can derive readily the connection one form corresponding to $\hat{H}_2(\vartheta, \varphi)$ defined in (3.11):

$$\mathbb{A}(\vartheta, \varphi) = \mathrm{i}\left[\left(\cos\vartheta\frac{\sigma_3}{2} + \alpha\sin\vartheta\frac{\sigma_1}{2}\right)\mathrm{d}\varphi + \alpha\frac{\sigma_2}{2}\mathrm{d}\vartheta\right]. \tag{4.2.4}$$

We would also remark that as $\Delta(\vartheta, \varphi)\cdot \mathrm{d}\boldsymbol{R} = \mathscr{A}(\vartheta, \varphi) - \mathbb{A}(\vartheta, \varphi)$, we can derive (4.2.4) with (4.6) and (4.2.3).

Furthermore we note that the gauge field $\mathbb{F}(\vartheta, \varphi)$ is given by:

$$\mathbb{F}(\vartheta, \varphi) = \mathrm{i}(\alpha^2 - 1)\frac{\sigma_3}{2}\mathrm{d}\Omega. \tag{4.2.5}$$

Comparing with (4.6) and (4.7), we find that the Berry's connection one forms and the gauge fields are related respectively by:

$$\mathbb{A}(\vartheta, \varphi) = \sigma_1 \mathscr{A}(\vartheta, \varphi)\sigma_1 \tag{4.2.6}$$

$$\mathbb{F}(\vartheta, \varphi) = \sigma_1 \mathscr{F}(\vartheta, \varphi)\sigma_1 \tag{4.2.7}$$

and vice versa since $\sigma_1^2 = I$. Because σ_1 is a Hermitian 2×2 unitary matrix belonging to U(2) and $d\sigma_1 = 0$, $\mathbb{A}(\vartheta, \varphi)$ and $\mathscr{A}(\vartheta, \varphi)$ are actually related by gauge transformation.

In addition, the invariant topological quantities defined in (3.23) and (3.24) are given by:

$$\mu_l(\vartheta, \varphi)\cdot \mathrm{d}(\vartheta, \varphi) = \lambda_l(\vartheta, \varphi)\cdot \mathrm{d}(\vartheta, \varphi) = \mathrm{i}\alpha \sin\vartheta\sigma_1\,\mathrm{d}\varphi \tag{4.2.8}$$

with $l = 1$ or 2. Again $\mu_l(\vartheta, \varphi) = \lambda_l(\vartheta, \varphi)$ because the instantaneous energy $E(\vartheta, \varphi)$ is independent on (ϑ, φ) as stated in section 4.1.

Furthermore, as discussed in [10], the connection one forms in (4.6) and (4.2.4) are well defined only on the equatorial patch and then denoted by $\mathscr{A}_E(\vartheta, \varphi)$ and $\mathbb{A}_E(\vartheta, \varphi)$ respectively. By gauge transformation with unitary matrix $\rho = \mathrm{e}^{-\mathrm{i}(\sigma_3/2)\varphi}$, we obtain a connection one form which is well defined on the south hemisphere of the 2-sphere:

$$\begin{aligned}\mathscr{A}_S(\vartheta, \varphi) &= \rho^+ \mathscr{A}_E(\vartheta, \varphi)\rho + \rho^+\,\mathrm{d}\rho \\ &= (-\mathrm{i})\left\{\left[(1+\cos\vartheta)\frac{\sigma_3}{2} - \alpha\sin\vartheta\left(\cos\varphi\frac{\sigma_1}{2} - \sin\varphi\frac{\sigma_2}{2}\right)\right]\mathrm{d}\varphi\right. \\ &\quad \left. + \alpha\left(\cos\varphi\frac{\sigma_2}{2} + \sin\varphi\frac{\sigma_1}{2}\right)\mathrm{d}\vartheta\right\}.\end{aligned} \tag{4.2.9}$$

Meanwhile, by gauge transformation of $\mathbb{A}_E(\vartheta, \varphi)$ with the same unitary matrix ρ, we yield a connection one form which is well defined on the north hemisphere and hence denoted by $\mathbb{A}_N(\vartheta, \varphi)$:

$$\begin{aligned}\mathbb{A}_N(\vartheta, \varphi) &= \rho^+ \mathbb{A}_E(\vartheta, \varphi)\rho + \rho^+\mathrm{d}\rho \\ &= \mathrm{i}\left\{\left[(-1+\cos\vartheta)\frac{\sigma_3}{2} + \alpha\sin\vartheta\left(\cos\varphi\frac{\sigma_1}{2} - \sin\varphi\frac{\sigma_2}{2}\right)\right]\mathrm{d}\varphi\right. \\ &\quad \left. + \alpha\left(\cos\varphi\frac{\sigma_2}{2} + \sin\varphi\frac{\sigma_1}{2}\right)\mathrm{d}\vartheta\right\}.\end{aligned} \tag{4.2.10}$$

The non-Abelian Berry's phase $\mathscr{U}$ (or $\mathbb{U}$) will be different if we use different connections $\mathscr{A}_E(\vartheta, \varphi)$ and $\mathscr{A}_S(\vartheta, \varphi)$ (or $\mathbb{A}_E(\vartheta, \varphi)$ and $\mathbb{A}_N(\vartheta, \varphi)$):

$$\begin{aligned}\mathscr{U}_E = \mathscr{P}\exp\left[-\int_1^2 \mathscr{A}_E(\vartheta, \varphi)\right] &= \rho(2)\left\{\mathscr{P}\exp\left[-\int_1^2 \mathscr{A}_S(\vartheta, \varphi)\right]\right\}\rho^+(1) \\ &= \rho(2)\mathscr{U}_S\rho^+(1)\end{aligned} \tag{4.2.11a}$$

$$\begin{aligned}\mathbb{U}_E = \mathscr{P}\exp\left[-\int_1^2 \mathbb{A}_E(\vartheta, \varphi)\right] &= \rho(2)\left\{\mathscr{P}\exp\left[-\int_1^2 \mathbb{A}_N(\vartheta, \varphi)\right]\right\}\rho^+(1) \\ &= \rho(2)\mathbb{U}_N\rho^+(1)\end{aligned} \tag{4.2.11b}$$

where 1 and 2 are the endings of a line segment on the 2-sphere. Moreover the non-Abelian Berry's phases $\mathscr{U}$ and $\mathbb{U}$ are related by:

$$\mathbb{U}_E = \sigma_1 \mathscr{U}_E \sigma_1 \tag{4.2.12a}$$

$$\mathbb{U}_N = \rho^2(2)\sigma_1 \mathscr{U}_S \sigma_1 (\rho^+(1))^2. \tag{4.2.12b}$$

In view of (4.2.6), we have found that the two supersymmetrically related systems are corresponding to different choices of gauge in the transformation specified by (3.25a, b). Finally it is worth noting that the invariant connection one form $\lambda_l(\vartheta, \varphi)\cdot \mathrm{d}(\vartheta, \varphi)$ given by (4.2.8) is not related to $\mathscr{A}_E(\vartheta, \varphi)$ (or $\mathbb{A}_E(\vartheta, \varphi)$) by gauge transformation.

5. Discussions

We have generalized our analysis on properties of Abelian Berry's phases in two supersymmetric related quantum systems [8] to that pertaining to systems whose Berry's phases are non-Abelian. In the Abelian case, we can derive an explicit expression for the difference in the Abelian Berry's connections of two supersymmetrically related quantum systems. Following, we can also obtain the difference between the corresponding Abelian Berry's phases. However, in the non-Abelian Berry's case, we have shown in section 3 that an explicit expression for the difference in the non-Abelian Berry's connections can be derived (stated in (3.14)) but we are not able to derive the difference between the non-Abelian Berry's phases. The reason is that: in the Abelian case, the gauge group is U(1) and the difference in the Berry's connections, say δ, determines the difference between the Berry's phases (i.e. $-\mathrm{i}\int \delta$). On the other hand, in the non-Abelian case, the gauge group is the non-Abelian group U(N) with $N>1$. The difference in the non-Abelian Berry's connections does not simply determine the difference between the non-Abelian Berry's phases because the phases themselves are not expressible in simple phase factor forms but are in forms of non-commutative unitary matrices.

Moreover, in either Abelian or non-Abelian case, a topological quantity which is invariant in the two supersymmetrically related quantum systems can be constructed. In both cases, the topological quantities are identical to connections one forms; moreover the holonomies corresponding to the connections [14] are obvious invariant topological 'phases' (both Abelian and non-Abelian) in the two supersymmetrically related quantum systems.

In section 5, we have discussed a quantum system described by the spin quadrupole Hamiltonian. We found that, in view of (4.8) and (4.10), there are infinite pairs of supersymmetric partners ($\hat{H}_1$ and $\hat{H}_2$) constructed by the factorization of supersymmetric quantum mechanics. Among different pairs of supersymmetric partners, we chose two particular cases to illustrate our results. In the first one, $\hat{H}_2$ (4.1.1) is chosen to be time independent and hence gives trivial gauge potential and field as stated in (4.1.2). The difference in the Berry's connections $\boldsymbol{\Delta}\cdot\mathrm{d}\boldsymbol{R}$ and the supersymmetric invariant connection $\boldsymbol{\lambda}\cdot\mathrm{d}\boldsymbol{R}$ are hence solely dependent on the Berry's connection of $\hat{H}_1$ as shown in (4.1.3, 4).

In the second case, $\hat{H}_2$ (4.2.1) is chosen to be a form very similar to $\hat{H}_1$. The orientations of the magnetic fields involved in the two supersymmetrically related systems are just opposite to each other in directions. We have derived the difference in the Berry's connections $\boldsymbol{\Delta}$ (4.2.3) and supersymmetric invariant connection $\boldsymbol{\lambda}$ (4.2.8). The Berry's connections $\mathscr{A}$ and $\mathbb{A}$ can be calculated by using instantaneous eigenstates given in (4.5) and (4.2.2) respectively. It is found that these connections are well defined only on the equatorial patch on the 2-sphere. Using gauge transformation with unitary matrix $\rho = e^{-i(\sigma_3/2)\varphi}$, they can be transformed to be connections $\mathscr{A}_S$ (4.2.9) and $\mathbb{A}_N$ (4.2.10) which are well defined on the south and north hemispheres respectively. In fact it is not possible to define a global connection which behaves well everywhere on the 2-sphere [15, 16]. Furthermore, we have found out the relations between the non-Abelian Berry's phases $\mathscr{U}$ and $\mathbb{U}$ as stated in (4.2.11, 12). In addition, it is worth mentioning that the supersymmetric invariant connection λ is not related to $\mathscr{A}$ (or $\mathbb{A}$) by gauge transformation. This point can be seen by considering a gauge covariant quantity $D_\varphi F_{\vartheta\varphi} = \partial_\varphi F_{\vartheta\varphi} + [A_\varphi, F_{\vartheta\varphi}]$, the source current of the gauge field [10]. Computed with λ, this quantity is zero but it is not zero when derived using $\mathscr{A}$ (or $\mathbb{A}$).

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